# ON THE POSTULATE OF PLASTICITY

## (O POSTULATE PLASTICHNOSTI)

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A. A. IL'IUSHIN
(Moscow)

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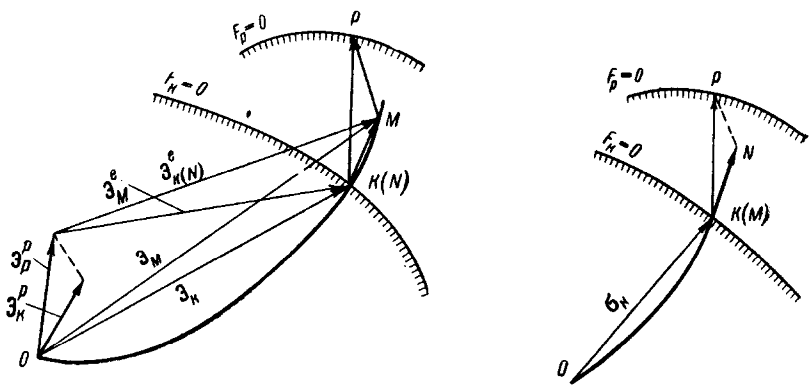
Let a body pass from a state of strain A into an infinitely close state of strain B (at constant temperature). The indication that this change of state is accompanied by plastic deformation is, by definition, the change of the plastic component of the total strain. This statement, appearing to be a tautology, actually contains the hypothesis of unloading. The states of stress and strain should be homogeneous; in the state A the plastic strain should be determined by unloading, and then the original loading should restore the state; after this, the loading should be changed to its value in the state B, and the unloading should be repeated in order to determine the change of the plastic component of strain. The hypothesis of unloading is valid only with certain accuracy and, therefore, the described criterion for increment of plastic strain during the change of state from A to B is approximate, as it is inherent to a physical criterion.

We can give also another definition and another criterion. Let us consider an arbitrary process AB; by changing the loading, we return the body from the state of strain B into the state of strain A, and thus we perform a closed cycle of deformation ABA; every intermediate state is assumed to be an equilibrium state. We shall consider that the transition from A to B is accompanied by plastic deformations if the work of external forces on the closed cycle ABA is positive, and it is purely elastic if the work is zero. This energetic definition we shall call the postulate of plasticity. It gives a definition of plastic deformation which is independent from the hypothesis of unloading, and it also indicates a specific method of experimental test.

There is no reason to reject any one of the above definitions of plastic strain and, therefore, they should be compatible. Hence, as we shall see, important consequences will be derived.

Both definitions confirm irreversibility of the process of plastic deformation. Some authors [1] are inclined to consider that the postulate of Drucker asserts the irreversibility of the process of plastic deformation. Drucker's postulate states that if the body passes from the state B into the state $A_0$ in which the loading returns to its original value (i.e. as in the state A), then the work of additional loading on this closed cycle of loading $ABA_0$ should be non-negative. But this work is equal to zero not only for an elastic process, but also for non-hardening solids and in some irreversible processes, for instance, in the extension of a specimen in the plastic flow range. This means that the Drucker's postulate does not specify the irreversibility of the process of plastic deformation, but is a special hypothesis.

We shall assume in the following that the reader is familiar with the papers [2,3,4]; we shall use the postulate of isotropy and the isotropic spaces of the deformation vector э and the stress vector $\sigma$. Let the process of deformation, at some instant of time, be determined by a progressing trajectory $OK$ and the point $K$, and let $F_K = 0$ be the equation of the yield surface in the deformation space and thus the points inside of this surface correspond to the possible states of unloading, and the points outside correspond to the states of increasing loading. By increasing loading, we move from the point $K$ to a nearby point $P$; the yield surface changes and its equation becomes $F_P = 0$. Figure 1a shows this process: the deformation $э_K$ and the plastic deformations $э_K^p$ and $э_P^p$.




a b

Fig. 1.

Figure 1b shows the same process in the space of stresses. Let the point $M$ in Fig. 1a correspond to unloading (i.e. the stress equal to the initial stress $\sigma_K$) and let the point $N$, coinciding with $K$, close the process in strain; in Fig. 1b the points $K$ and $M$ coincide. The examination of Figs. 1a and 1b reveals that the points $M$ and $N$ are located inside the surface $F_P$, and therefore the work of the total stress $\sigma$ on the path $KPK$ (Fig. 1a) is equal to the work along the path $KPMK$. Let us denote the work by $W$:

$$W = \int_{KPK} \sigma \, d\text{э} = \int_{KPMK} \sigma \, d\text{э} \tag{1}$$

Since the work of the constant stress $\sigma_K$ on this path is equal to zero, it follows that

$$W = \int_{KPMK} (\sigma - \sigma_K) \, d\text{э}$$

The part of this integral on the path $KPM$ represents the work of additional stresses, which enter into Drucker's postulate. We denote this work by $W_D$ and obtain

$$W - W_D = \int_{MK} (\sigma - \sigma_K) \, d\text{э}, \qquad W_D = \int_{KPM} (\sigma - \sigma_K) \, d\text{э} \tag{2}$$

According to the definition of the point $M$, we have $\sigma_K = \sigma_M$ along the path $MP$, and since only the elastic strain changes, it is $d\text{э} = d(\text{э}^e - \overset{e}{\text{э}}_M)$. Considering the law of elasticity for the path $MP$, we have

$$\sigma - \sigma_M = (E)_P (\text{э}^e - \overset{e}{\text{э}}_M)$$

where $(E)_P$ is the matrix of the moduli of elasticity with respect to the point $P$. We conclude then that the difference

$$W - W_D = \int_{MK} (E)_P \, (\text{э}^e - \overset{e}{\text{э}}_M) \, d\, (\text{э}^e - \overset{e}{\text{э}}_M) > 0$$

represents the elastic strain energy corresponding to the difference of elastic deformations $\overset{e}{\text{э}}_N - \overset{e}{\text{э}}_M$, i.e. a positive quadratic form. This means that $W > W_D$. Furthermore, this indicates that the postulate of plasticity is more general, less restrictive, than Drucker's postulate, and that the latter is a sufficient but not necessary condition in the framework of the former.

We shall consider now some essential consequences of the two definitions, given above, of the process of plastic deformation without the use of Drucker's postulate. Figure 2 shows again the process $OK$ in the space of deformation, the surface of flow $F_K = 0$, and the vector of plastic deformation $\text{э}_K^p$ for $K$. By way of unloading, we move the body into the state represented by the point $M$, and then we perform a process $MTPTM$ closed with respect to deformation. The point $T$ is located on the surface $F_K$ and is determined by the length $x$ of the segment $MT$ and the unit vector $\mathbf{t}$; the point $P$ is determined by a small segment $\xi$ and the unit vector **e** in the region of additional loading. The vector of plastic deformation then becomes $\mathbf{э}_P^p$, and the yield surface is $F_P = 0$. The change of the vector of plastic deformations is denoted by

$$\Delta \mathbf{э}^p = \mathbf{э}_P^p - \mathbf{э}_K^p = \zeta \mathbf{p} \tag{3}$$

with **p** being a unit vector.

On the basis of the postulate of plasticity we have

$$W = \int\limits_{MTPTM} \sigma d\mathbf{э} \geqslant 0 \tag{4}$$

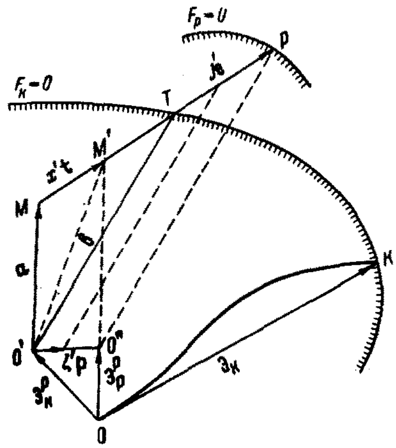


Fig. 2.

As the quantity $\xi$ is small, the matrix of elastic moduli $(E)$ changes linearly with $\xi$ in the transition from $T$ to $P$, and therefore, at any intermediate point $M$ and at the point $P$, it is

$$(E)_{\xi'} = (E)_T + \xi' \left[\frac{d}{d\xi}(E)\right]_T, \qquad (E)_P = (E)_T + \xi \left[\frac{d}{d\xi}(E)\right]_T \tag{5}$$

with, obviously, $(E)_T = (E)_K$. The sum of the integrals (4) along the paths $TP$ and $PT$

$$\int\limits_{TP} \sigma d\mathbf{э} + \int\limits_{PT} \sigma d\mathbf{э} = \int\limits_0^\xi (\sigma_{TP} - \sigma_{PT})\, \mathbf{e}\, d\xi'$$

is, on the basis of (5), a small quantity of the order $\xi^2$. Retaining in (4) quantities of the order $\xi$ only, we obtain

$$W = \int\limits_0^x (\sigma_{MT} - \sigma_{TM})\, \mathbf{t}\, dx'$$

where $\sigma_{MT}$ is to be calculated from the elastic deformation $\mathbf{a} + x'\mathbf{t}$ and with the matrix of elastic moduli $(E)_T$, while $\sigma_{TM}$ is to be calculated from the elastic deformation $\mathbf{a} + x'\mathbf{t} - \zeta\mathbf{p}$ and with the matrix $(E)_P$. We thus have

$$\sigma_{MT} - \sigma_{TM} = \zeta (E)_T \mathbf{p} - \xi \left[\frac{d}{d\xi}(E)\right]_T \{\mathbf{b} - (x - x')\,\mathbf{t}\}$$

and the work is equal to

$$W = x\,[\zeta\,(E)_T\,\mathbf{p} + \xi\,(E)_T{}'\mathbf{b}]\cdot\mathbf{t} + \frac{1}{2}\,x^2\xi\mathbf{t}\,(E)_T{}'\cdot\mathbf{t} \tag{6}$$

On the basis of (3) and considering that $\mathbf{b}$ is the elastic deformation at the point $T$ (at the crossing of the flow surface $F_K$), we transform (6) into the form

$$W = x\xi\boldsymbol{\sigma}^\circ\cdot\mathbf{t} + \frac{1}{2}\,x^2\xi\mathbf{t}\,(E)'\cdot\mathbf{t} \tag{7}$$

where the notation for a new physical vector is introduced

$$\boldsymbol{\sigma}^\circ = (E)\,\frac{d\mathbf{э}^p}{d\xi} - (E)'\,\mathbf{э}^e$$

and $(E)'$ is the derivative of the matrix $(E)$ with respect to the deformation $\xi$ in the direction $\mathbf{e}$.

Since, obviously

$$(E)\,\frac{d\mathbf{э}^e}{d\xi} + (E)'\,\mathbf{э}^e = \frac{d\boldsymbol{\sigma}}{d\xi}$$

the derivative of the matrix can be eliminated from the expression for $\boldsymbol{\sigma}^\circ$, and we thus obtain

$$\boldsymbol{\sigma}^\circ = (E)\,\frac{d\mathbf{э}}{d\xi} - \frac{d\boldsymbol{\sigma}}{d\xi} \tag{8}$$

This vector, evidently, does not depend on the position of the point $M$, but only on $T$ and the direction $TP$. Let us assume a point $M$ in the vicinity of $T$ on the surface $F_K = 0$ (approaching it from inside); $\mathbf{t}$ thus becomes a vector in the hyperplane tangent to $F_K$. From the postulate of plasticity and from (7) we obtain $\boldsymbol{\sigma}^\circ \cdot \mathbf{t} \geqslant 0$ for any vector $\mathbf{t}$. This is possible only in the case if $\boldsymbol{\sigma}^\circ \cdot \mathbf{t} = 0$. Thus the physical vector $\boldsymbol{\sigma}^\circ$ is normal to the flow surface at the point $T$, i.e.

$$\boldsymbol{\sigma}^\circ = D\,\mathrm{grad}\,F_K \tag{9}$$

Let us consider now a two-dimensional plane normal to $F_K$ and containing the vectors $\boldsymbol{\sigma}^\circ$ and $\mathbf{t}$; it intersects the flow surface along a line. The distance from this line to the tangent hyperplane (measured in the mentioned two-dimensional plane) can be expressed in terms of the distance $x$ and the curvature $\kappa$ of the line in the form $Z = 1/2\,\kappa x^2$, where $\kappa$ is considered to be positive if the line is concave. In the second approximation, the scalar product $\boldsymbol{\sigma}^\circ \cdot \mathbf{t}$ is equal to $-1/2\,\kappa x|\boldsymbol{\sigma}^\circ|$, and therefore we obtain from (7)

$$W = \frac{1}{2} x^2 \xi [-\varkappa |\boldsymbol{\sigma}^\circ| + \mathbf{t}(E)'\mathbf{t}]$$

Since $\xi > 0$, we obtain from $W > 0$

$$-\varkappa |\boldsymbol{\sigma}^\circ| + \mathbf{t}(E)'\mathbf{t} > 0 \tag{10}$$

This, in general, does not imply the convexity of the surface $F_K$, although convex surfaces satisfy this condition..

The assumption of the postulate of plasticity for an analogous process in the space of stresses leads to the following analogous results. The physical vector

$$\mathbf{э}^\circ \equiv \frac{d\mathbf{э}^p}{d\Sigma} + (\mathscr{E})'_\Sigma \boldsymbol{\sigma} \equiv \frac{d\mathbf{э}}{d\Sigma} - (\mathscr{E}) \frac{d\boldsymbol{\sigma}}{d\Sigma} = G^\circ \operatorname{grad} f_K \tag{11}$$

is normal to the yield surface in the space of stresses $f_K = 0$. Here, $d\Sigma = |d\boldsymbol{\sigma}|$ and $(\mathscr{E}) = (E)^{-1}$.

If a trajectory in the form of a polygonal line is considered [3], the substitution

$$G^\circ d\Sigma = G df \tag{12}$$

is introduced into (11), the expressions (5.1) and (5.10) of [3] are used, and considering that in the case discussed

$$\operatorname{grad} f_K = |\operatorname{grad} f_K| \frac{\boldsymbol{\sigma}}{\sigma}$$

we obtain

$$N = N_K, \quad \frac{1}{P} - \frac{1}{P_K} = G (\operatorname{grad} f_K)^2 \tag{13}$$

In conclusion, let us summarize the obtained results. New relations between the physical vectors $\boldsymbol{\sigma}^\circ$ and $\mathbf{э}^\circ$ and the yield surfaces $F_K$ and $f_K$ are given. One of these relations, for the particular case of absence of deformational anisotropy, was obtained earlier by Drucker from his postulate. Now it has been shown that the results of Drucker and the new relations are the consequences of more general plastic properties of solids than those compatible with Drucker's postulate.

BIBLIOGRAPHY

1. Goodier, J.N. and Hodge, P.G., Jr., *Uprugost' i plastichnost' (Elasticity and Plasticity)*, p. 80. IIL, 1960.

2. Il'iushin, A.A., O sviazi mezhdu napriazheniami i malymi deformatsiiami v mekhanike sploshnykh sred (On the relations between stresses and small deformations in mechanics of continuous media). *PMM* Vol. 18, No. 6, 1954.

3. Il'iushin, A.A., Voprosy obshchei teorii plastichnosti (Problems of the general theory of plasticity). *PMM* Vol. 24, No. 3, 1960.

4. Il'iushin, A.A., O prirashchenii plasticheskoi deformatsii i poverkhnosti tekuchesti (On the increments of plastic deformations and the yield surface). *PMM* Vol. 24, No. 4, 1960.

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